Systemic Bank Panics in Financial Networks

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Purple comments reflect the views of only the presenter (James Siderius) and not of the author (Zhen Zhou)

Why am I presenting this paper?

- The topic:
  - relevant to the 2008 financial crisis
  - first paper to explain how financial linkages trigger financial crises *without fundamentals*
  - "panics" are a big source of problems today
- The paper:
  - skillfully navigates an ambitious (and thought to be intractable) setup
  - proofs are well-written and offer helpful techniques
  - helps become more comfortable with global games
- Today:
  - global games refresher: bank runs
  - model
  - proofs of equilibrium characterization
  - key results

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Bank Runs: Diamond and Dybvig (1983)

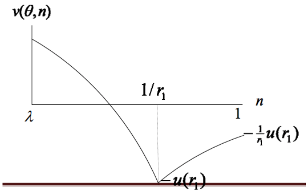
- Continuum of depositors on $[0,2]$ who deposit \$1 in the bank at $t=0$. There are three time periods $t=0,1,2$.
- At $t=1$, a preference shock is realized for each depositor; she is either *patient* or *impatient* with equal probability.
- Bank invests in a project of cost $\theta > 0$ that yields a deterministic payoff $\theta(1+r)$ at $t=2$ but has zero liquidation value at $t=1$.
- If she is impatient, she withdraws immediately; if she is patient, she values consumption at both $t=1$ and $t=2$.
  - If she withdraws at $t=1$ ("withdraw early") she receives:
  $$\min\left\{1, \frac{2-\theta}{1+w}\right\}$$
  where w is the fraction of (patient) depositors who withdraw early.
  - If she withdraws at $t=2$ ("delay withdrawal") she receives nothing if $1+w > 2-\theta$ (the bank went insolvent) and otherwise gets $1+r$.
  - If $w > 1-\theta$, depositor i prefers to withdraw early; if $w < 1-\theta$, depositor i prefers to wait.
- The game features strategic complementarities: if others withdraw early, you are better off withdrawing early, and vice-versa. There is a **good** and **bad** equilibrium, depending on whether depositors "panic".

Bank Runs: Goldstein and Pauzner (2005)

- Well, which is it?
- Still three periods $t = 0, 1, 2$ with a continuum $[0, 1]$ of agents who are with equal probability either patient or impatient (realized at $t = 1$) and deposit \$1 at $t = 0$.
- Nature picks some parameter $\theta \sim U[0, 1]$. The bank holds a risky asset that pays out either 0 or R, with probability $p(\theta)$ and $1 - p(\theta)$, respectively, in $t = 2$ but is worthless at $t = 1$. Moreover, $p(\theta)$ is increasing in θ.
- Let w denote the fraction of (all) depositors who withdraw early.
- If a depositor withdraws early, she receives r_1 assuming that $w < 1/r_1$ (i.e., the bank can meet its obligations at $t = 1$). If $w > 1/r_1$, then depositors are "randomly repaid" with probability $\frac{1}{wr_1}$.
- In the former case, those who delayed withdrawing have claims to the asset and thus receive $\frac{1-wr_1}{1-w}R$ with probability $p(\theta)$ and nothing with probability $1 - p(\theta)$. In the latter case, the bank went insolvent, so those withdrawing late receive nothing.
- **Key modeling step**: At $t = 0$, agents receive a private signal about the strength of the bank, $\theta_i = \theta + \varepsilon_i$, with ε_i iid across depositors.

Goldstein and Pauzner (2005)

- This is a global game. **The goal**: There exists a unique cutoff equilibrium θ^* where agents with $\theta_i > \theta^*$ delay withdrawing and those with $\theta_i < \theta^*$ withdraw early.
- *Usual proof technique*: (i) Show there are upper and lower dominance regions, i.e., $\underline{\theta}$ and $\bar{\theta}$ such that if $\theta > \bar{\theta}$, it is strictly dominant to delay and if $\theta < \underline{\theta}$ it is strictly dominant to withdraw early, and (ii) the game is of global complementarities (i.e., difference in utility of action A and B increases as more do action A).
- This one is not; withdrawing becomes "less appealing" after $w > 1/r_1$:



- The proof technique can be amended by arguing that the payoff is monotonically decreasing wherever it is positive, and so still has a single crossing and unique cutoff.

Model: Preliminaries

- Single-product economy with n banks.
- Each bank has risk-neutral creditors of measure 2, and each creditor is only associated with one bank.
- The economy lasts for three periods, $t = 0, 1, 2$ (clearer with four periods $t = 0, 1a, 1b, 2$):
  - Each creditor is endowed with one unit of capital and deposits this into the bank at $t = 0$.
  - The deposit contract allows the creditor to make withdrawals at either $t = 1a$ or $t = 2$, but creditors will have uncertainty about their liquidity needs.
  - A preference shock is realized at $t = 1a$, the creditor is either:
    - *Impatient*, and only values consumption at $t = 1b$;
    - *Patient*, and values consumption equally at $t = 1b$ and $t = 2$.
  - The patient creditor makes a withdrawal decision at $t = 1a$, denoted by a_{im} for bank i and creditor m, with $a_{im} = 0$ meaning early withdrawal (at $t = 1b$) and $a_{im} = 1$ meaning late withdrawal ($t = 2$).
  - Then $w_i = \int_0^1 \mathbf{1}(a_{im} = 0)\, dm$ denotes the share of bank i's patient creditors who decide to withdraw early.

Model: Assets and Liabilities

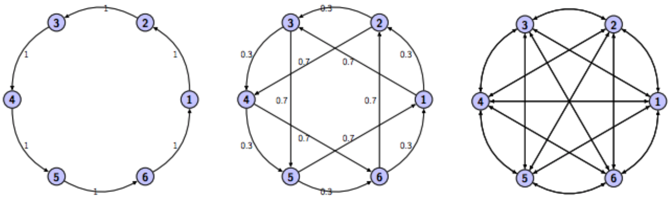
- After receiving the (two units) of capital from the creditors, the bank makes a long-term investment A_i which will be realized at $t = 2$ and is *riskless*.
  - **Assumption**: This investment has enough return to compensate all the patient creditors who did not withdraw in $t = 1a$.
- Bank i also has a "legacy asset" which generates a cash flow of $1 + \theta_i$ at $t = 1b$.
  - θ_i is random, so can be interpreted as the liquidity shock to bank i. It is realized at $t = 1b$, *after* the withdrawal decision at $t = 1a$.
  - Neither asset is pledgeable at $t = 1b$, and if the bank fails to meet its obligations at $t = 1b$, both the long-term investment and the legacy asset go through a costly liquidation process, and nothing is recovered.
  - This seems like a typo in the paper – it is just the long-term investment that is not pledgeable.
  - We call a bank "insolvent" when this liquidation takes place.
- Banks also build financial connections through unsecured debt contracts.
  - k_{ij} is the amount of capital borrowed by bank j from bank i.
  - At $t = 1b$, bank j needs to repay $y_{ij} = R_{ij}k_{ij}$ back to bank i (note here that R_{ij} is the corresponding interest rate).
  - We let $x_{ij} \leq y_{ij}$ denote the repayment that bank j makes to bank i at $t = 1b$.
  - Note that if bank j does not have sufficient liquidity at $t = 1b$ to fulfill its liabilities at $t = 1b$, then $x_{ij} < y_{ij}$.

Model: Financial Networks

- We take the financial connections that form the financial network as given. The pair $(\mathbf{Q}_{n\times n}, \mathbf{y})$ gives the liabilities matrix $\mathbf{Q}$:

$$\mathbf{Q}_{ij} = \begin{cases} \frac{y_{ij}}{y_j}, \text{ if } y_j > 0 \\ 0, \text{ otherwise} \end{cases}$$

  and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ is the vector of interbank liabilities.
- A financial network is *regular* if each bank has identical interbank liabilities and claims, i.e., $\sum_{i\neq j} y_{ij} = \sum_{j\neq i} y_{ji} = y$.
- A financial network is *symmetric* if and only for all banks i, j, and all $k \in \{1, \ldots, n-1\}$, $\mathbf{Q}_{i,i+k \mod n} = \mathbf{Q}_{j,j+k \mod n}$.



Model: Repayments

- Non-bank creditors who withdraw early and creditor banks have claims on the liquid assets of bank i at $t = 1b$, but non-bank creditors are more senior.
- A bank defaults on its senior liabilities if $1 + \theta_i + \sum_{j \neq i} x_{ij} < 1 + w_i$; if so, no creditor banks get repaid and all non-bank creditors who withdraw are repaid evenly at $t = 1b$.
- If a bank defaults only on its interbank loans, i.e., $1 + w_i \leq 1 + \theta_i + \sum_{j \neq i} x_{ij} < 1 + w_i + y_i$, the withdrawals are met in full (at $t = 1b$) and the creditor banks are repaid in proportion to the face value of the interbank loans. In other words:

$$x_{ij} = \frac{y_{ij}}{y_j} \left[\min\{\theta_j - w_j + \sum_{i \neq k} x_{ki}, y_j\} \right]^+$$

- Define $e_i \equiv \theta_i - w_i$ as bank i's residual liquidity after meeting senior creditors' withdrawals. Following the notation of Eisenberg and Noe (2001):

Definition

The *clearing payment vector* is a fixed point of the mapping $\Phi(\mathbf{x}|\mathbf{Q}, \mathbf{y}, \mathbf{e}) : \prod_{i=1}^{n} [0, y_i] \to \prod_{i=1}^{n} [0, y_i]$ satisfying:

$$\Phi(\mathbf{x}|\mathbf{Q}, \mathbf{y}, \mathbf{e}) = [\min\{\mathbf{e} + \mathbf{Q}\mathbf{x}, \mathbf{y}\}]^+$$

Model: Strategic Withdrawals

- If you are a patient creditor, you must decide whether to withdraw or not. You know the financial network, but have incomplete information about bank i's liquidity shock θ_i.
- Nature picks θ_j from the uniform prior $[\underline{\theta}, \overline{\theta}]$.
- Each creditor gets noisy private information about this liquidity shock, $s_{im} = \theta_i + \sigma\epsilon_{im}$ where the error term is iid on $U[-1/2, 1/2]$ across creditors.
- **Assumption**: $\bar{\theta} > 1 + \sigma$ and $\underline{\theta} < -\sigma$:
  1. If $s_i \in [-\sigma/2, 1 + \sigma/2]$, the liquidity θ_i is uniformly distributed on $[s_i - \frac{1}{2}\sigma, s_i + \frac{1}{2}\sigma]$;
  2. If $s_i < -\sigma/2$, then $\theta_i < 0$;
  3. If $s_i > 1 + \sigma/2$, then $\theta_i > 1$.
- If the bank is solvent at $t = 2$, a patient creditor who delays $a_{im} = 1$ receives $1 + r$. If the bank defaults at $t = 1$, creditors who withdraw early split the bank's liquid assets (up to principal value 1) and all remaining creditors get nothing. Formally:

$$u(a_{im} = 0, w_i, \theta_i, \mathbf{Q}, \mathbf{y}) = \begin{cases} 1, \text{ if } e_i + \sum_{j \neq i} x_{ij} \geq y_i \\ \min\left\{1, \frac{\theta_i + \sum_{k \neq i} x_{ik} + 1}{w_i + 1}\right\}, \text{ if } e_i + \sum_{j \neq i} x_{ij} < y_i \end{cases}$$

$$u(a_{im} = 1, w_i, \theta_i, \mathbf{Q}, \mathbf{y}) = \begin{cases} 1 + r, \text{ if } e_i + \sum_{k \neq i} x_{ik} \geq y_i \\ 0, \text{ if } e_i + \sum_{k \neq i} x_{ik} < y_i \end{cases}$$

Equilibrium

Definition

For a given financial network $(\mathbf{Q}, \mathbf{y})$, the Bayesian Nash equilibrium is defined as a collection of n withdrawal sets $\{\mathbb{S}_i\}_{i=1}^n$. Patient creditors of bank i will withdraw early if and only if they receive private information $s_i \in \mathbb{S}_i$. The Bayesian Nash equilibrium $\mathcal{S} \equiv \{\mathbb{S}_i\}_{i=1}^n$ satisfies the following conditions:

1. For given realizations $\boldsymbol{\theta} \equiv (\theta_1, \ldots, \theta_n)^T$ and all other creditors' strategies $\mathcal{S}$ such that $e_i = \theta_i - w_i = \theta_i - \mathbb{P}(s_i \in \mathbb{S}_i | \theta_i)$, the vector $\mathbf{x}(\mathbf{e}(\mathbb{S}, \boldsymbol{\theta}), \mathbf{Q}, \mathbf{y})$ is the clearing payment vector.
2. For a given $\mathcal{S}$, for all $i \in \{1, 2, \ldots, n\}$ and all $s_i \in \mathbb{S}_i$, the expected payoff difference is

$$H_i(s_i, \mathcal{S}) = \int_{\boldsymbol{\theta} \in \{\boldsymbol{\theta} | x_i(\mathbf{e}(\mathcal{S}, \boldsymbol{\theta}), \mathbf{Q}, \mathbf{y}) = y\}} r dF(\boldsymbol{\theta}|s_i) + \int_{\boldsymbol{\theta} \in \boldsymbol{\theta} | 0 < x_i < y} (-1) dF(\boldsymbol{\theta}|s_i)$$
$$+ \int_{\boldsymbol{\theta} \in \{\boldsymbol{\theta} | x_i = 0\}} \left(-\frac{\theta_i + (\mathbf{Qx})_i + 1}{w_i + 1} \right) dF(\boldsymbol{\theta}|s_i) < 0,$$

   where $F(\boldsymbol{\theta}|s_i)$ denotes the CDF of $\boldsymbol{\theta}$, given s_i. Moreover, for all $s_i \in \mathbb{S}_i^C$, $H_i(s_i, \mathcal{S}) \geq 0$.

Existence and Generic Uniqueness of Payment Clearing

Lemma

When $\theta_i \sim U[\underline{\theta}, \overline{\theta}]$, the clearing payment vector $\mathbf{x}(\mathbf{e}, \mathbf{Q}, \mathbf{y})$ exists and is generically unique (almost surely?).

Proof Sketch. Existence follows from Acemoglu et al (2015), but more simply checking that one can apply Brouwer's fixed point theorem. It is shown in Acemoglu et al (2015) that:

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i + \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} x_j$$

if there exist two distinct clearing vectors $\mathbf{x}, \hat{\mathbf{x}}$.

- Intuition: if there exists a bank not in default, then $\Phi(\mathbf{x}|\mathbf{Q}, \mathbf{y}, \mathbf{e})$ is a contraction map because $|[\min\{\alpha, \beta\}]^+ - [\min\{\hat{\alpha}, \beta\}]^+| < |\alpha - \hat{\alpha}|$ provided that $\alpha \neq \alpha'$ *and* either $\alpha \notin [0, \beta]$ or $\hat{\alpha} \notin [0, \beta]$, so the clearing vector must be unique.

But since $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} x_j$, it must be that $\sum_{i=1}^{n} e_i = 0$. Note that $\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (\theta_i - w_i) = \sum_{i=1}^{n} \theta_i - \mathbb{P}[s_i \in \mathbb{S}_i | \theta_i] = \sum_{i=1}^{n} \theta_i - \mathbb{P}[\theta_i + \sigma \epsilon_{im} \in \mathbb{S}_i]$.

Assertion. $\sum_{i=1}^{n} \theta_i - \mathbb{P}[\theta_i + \sigma \epsilon_{im} \in \mathbb{S}_i]$ is continuously distributed.

Is this obvious?

- Does there exist some $\mathbb{S}_i$ such that $\mathbb{P}[\theta_i + \sigma\epsilon_{im} \in \mathbb{S}_i] = \theta_i$ for all $\theta_i \in (\theta_1, \theta_2)$?
- Let's suppose $\mathbb{S}_i = (s_1, s_2)$. Then:

$$\mathbb{P}[\theta_i + \sigma\epsilon_{im} \in (s_1, s_2)] = \min\left\{\frac{s_2 - \theta_i}{\sigma}, \frac{1}{2}\right\} - \max\left\{\frac{s_1 - \theta_i}{\sigma}, -\frac{1}{2}\right\}$$

  Take $\theta_1 = 0$ and $\theta_2 = \sigma - \delta$ for some small $\delta > 0$. Let $s_1 = \sigma/2$ and $s_2 > \frac{3\sigma}{2} - \delta$. Then for all $\theta_i \in (\theta_1, \theta_2)$:

$$\begin{aligned}\min\left\{\frac{s_2 - \theta_i}{\sigma}, \frac{1}{2}\right\} - \max\left\{\frac{s_1 - \theta_i}{\sigma}, -\frac{1}{2}\right\} &= \frac{1}{2} - \frac{s_1 - \theta_i}{\sigma} \\ &= \frac{\theta_i}{\sigma}\end{aligned}$$

- What if we let $\sigma = 1$? The fraction withdrawing increases one-for-one with increases in θ_i.
- Fine; rule out $\sigma = 1$. But for general σ, it is not obvious to me how to prove there exists no $\mathbb{S}_i$ where $\mathbb{P}[\theta_i + \sigma\epsilon_{im} \in \mathbb{S}_i] = \theta_i$ holds on a set of positive measure.
- *Comment*: Alternative approach is to show almost sure uniqueness of clearing payment vector not independent of $\mathbb{S}_i$, but given the equilibrium withdrawals. Under a threshold strategy, withdrawals must be decreasing in θ_i, so $\sum_i e_i$ is monotone in θ_i.

Equilibrium characterization: Global games cutoffs

Lemma

Let $(\mathbf{Q}, \mathbf{y})$ be a regular financial network. If $\sigma \leq \sigma_0 \equiv \frac{1+r}{\log 2} - 1$ and $y \leq y_0 \equiv \min\{-\underline{\theta}, \overline{\theta} - 1\}$, then for any given strategy profile of other banks' creditors $\{\mathbb{S}_{j \neq i}\}$, there exists a unique threshold $s_i^*(\{\mathbb{S}_{j \neq i}\}) \in [\underline{s}, \overline{s}]$ such that the only rationalizable strategy for bank i's creditor m is to choose $a_{im} = 1$ if and only if $s_i \geq s_i^*(\{\mathbb{S}_{j \neq i}\})$.

- *General technique*: Establish upper (never withdraw, even if all others do) and lower (always withdraw, even if no others do) dominance regions. Show the iterated elimination of dominated strategies can be performed from both the sides.
- Worst-possible case for creditors' withdrawal decisions: $\mathbb{S}_i = [\underline{s}, \bar{s}]$. Thus, $w_i = 1$ and $e_i = \theta_i - 1$.
- Worst possible repayment of interbank loans: $\sum_{j \neq i} x_{ij} = 0$. The bank's liquidity is $\theta_i + 1$ and its obligation is $2 + y$ (all creditors, patient or impatient, and interbank loans).
- If a creditor receives information $s_i \in [1 + y + \sigma/2, \bar{s}] \neq \emptyset$, then $\theta_i > 1 + y$, so will always be able to fulfill its obligations at $t = 1$; creditor will never withdraw.

Lemma 2 proof sketch, cont.

- Now, consider the best-case scenario for withdrawal decisions, $\mathbb{S}_i = \varnothing$ and the bank gets entirely repaid $y = \sum_{j \neq i} x_{ij}$.
- If a creditor receives $s_i \in [\underline{s}, -\sigma/2) \neq \varnothing$, she understands that $\theta_i < 0$ and the liquid assets held by the bank are at most $1 + \theta_i + y$. At the same time, the bank necessarily has at least outstanding liability of $1 + y$ (interbank loans and impatient creditors).
- Therefore, any creditor with $s_i < -\sigma/2$ will choose to withdraw regardless of the decisions of other creditors.
- If we continue with iterated elimination of dominated strategies, we see there are two thresholds, $\bar{s}_i$ and $\underline{s}_i$, such that the only rationalizable strategy when $s_i \geq \bar{s}_i$ is to not withdraw, whereas the only rationalizable strategy when $s_i \leq \underline{s}_i$ is to always withdraw.
- If $\underline{s}_i < \bar{s}_i$, there are at least two monotone equilibria (and possibly others).
  - There is a "worst" equilibrium where $s_i < \bar{s}_i$ always withdraw and a "best" equilibrium where $s_i > \underline{s}_i$ never withdraw.
- If we prove there exists a unique *monotone* equilibrium we have shown $\underline{s}_i = \bar{s}_i$:
  - Why? If these cutoffs are distinct, there must be *at least* two monotone equilibria.

Lemma 2 proof sketch, cont.

- Creditors (of the same bank) must play the same strategy in any equilibrium and because we have assumed it is monotone, there is a cutoff s^* such that for $s < s^*$ the creditor withdraws and if $s > s^*$ the creditor does not.
- Hold fixed the strategies $\{\mathbb{S}_j\}_{j \neq i}$ of creditors to other banks (which may or may not be in cutoff strategies).
- Let s_{im} be the signal creditor m gets about the liquidity of bank i and let $s_i^*(\{\mathbb{S}_j\}_{j \neq i})$ denote an arbitrary cutoff for bank i.
- Let $H(s_{im}, s_i^*)$ denote the difference in expected payoff between withdrawing and not given private signal s_{im}. We want to show that $H(s_{im}, s_i^*)$ admits a unique solution to $H(s_i^*, s_i^*) = 0$.
- Write:

$$H(s_{im}, s_i^*) = \int_{\boldsymbol{\theta}_{-i}} V(s_{im}, s_i^*) dJ(\boldsymbol{\theta}_{-i} | s_{im})$$

where $\boldsymbol{\theta}_{-i}$ is the realization of $\{\theta_j\}_{j \neq i}$ and $J(\boldsymbol{\theta}_{-i} | s_{im})$ (not independent?) denotes the CDF of $\boldsymbol{\theta}_{-i}$, given s_{im}, and $V(s_{im}, s_i^*)$ is the payoff difference from withdrawing or not for a creditor with information s_{im}, given $\boldsymbol{\theta}_{-i}$, and the monotone cutoff strategy s_i^* of other creditors in bank i.

Lemma 2 proof sketch, cont.

- Note that $V(s_{im}, s_i^*)$ is a function of $\sum_{j \neq i} x_{ij}$, so depends on how much bank i can pay to its creditor banks, which is monotone in θ_i.
  - Define $\theta_i^*(\boldsymbol{\theta}_{-i}, s_i^*)$ as the *junior liquidity threshold* (different term in paper) at which it can exactly fulfill its (junior+senior) obligations.
  - Define $\theta_i^0(\boldsymbol{\theta}_{-i}, s_i^*)$ as the *senior liquidity threshold* (again, different term in paper) at which it can exactly fulfill its senior obligations.
- Formally:

$$\theta_i^* + \sum_{j \neq i} x_{ij}(\theta_i^*) = w_i(\theta_i^*) + y$$

$$\theta_i^0 + \sum_{j \neq i} x_{ij}(\theta_i^0) = w_i(\theta_i^0)$$

- We want to rewrite V in terms of w_i:

$$V(s_i^*, s_i^*) = \frac{1+r}{1+\sigma}\left(\sum_{j \neq i} x_{ij}(w_i^*) - \sum_{j \neq i} x_{ij}(w_i^0) - y\right) + \frac{r-\sigma}{1+\sigma}\left(s_i^* + \frac{\sigma}{2} + \sum_{j \neq i} x_{ij}(w_i^0)\right)$$
$$- \left(s_i^* + \frac{\sigma}{2} + \sum_{j \neq i} x_{ij}(w_i^0) + 1 + \sigma\right)\left[\log(2) - \log\left(\frac{s_i^* + \sigma/2 + \sum_{j \neq i} x_{ij}(w_i^0)}{1+\sigma} + 1\right)\right]$$

where w_i^* and w_i^0 are the aggregate withdrawals when $\theta_i = \theta_i^*$ and $\theta_i = \theta_i^0$, respectively.

Lemma 2 proof sketch, cont.

- Let us simply rewrite this as:

$$H(s_i^*, s_i^*) = \int_{\boldsymbol{\theta}_{-i}} \left(\frac{1+r}{1+\sigma} \left(\sum_{j \neq i} x_{ij}(w_i^*) - \sum_{j \neq i} x_{ij}(w_i^0) - y \right) + U(s_i^*, s_i^*) \right) dJ(\boldsymbol{\theta}_{-i} | s_{im})$$

  where $U(s_i^*, s_i^*)$ is some expression in terms of $\sum_{j \neq i} x_{ij}(w_i^0)$ and other model primitives.

- **Claim:** The term $\int_{\boldsymbol{\theta}_{-i}} \frac{1+r}{1+\sigma} \left(\sum_{j \neq i} x_{ij}(w_i^*) - \sum_{j \neq i} x_{ij}(w_i^0) - y \right) dJ(\boldsymbol{\theta}_{-i} | s_{im})$ only depends on the financial network $(\mathbf{Q}, \mathbf{y})$ (and not the cutoff s_i^*).
  - If bank i receives $\sum_j x_{ij}(w_i^*)$, under withdrawal strategy s_i^*, bank i repays exactly y to all creditor banks.
  - If bank i receives $\sum_j x_{ij}(w_i^0)$, bank i defaults and so repays nothing to its creditor banks.
  - If bank i receives something in-between, let's "reimburse" all other banks $j \neq i$ the difference to *their own repayment* between $x_i = 0$ and $x_i = y$ through θ_j.
  - This leaves the interbank payment vector $\mathbf{x}$ unchanged.

Reimbursement step of Lemma 2

- **Claim.** It is the case that $\sum_{j \neq i} x_{ij}(w_i^*)$ with $\theta_j = \theta_j'$ is the same as $\sum_{j \neq i} x_{ij}(w_i^0)$ when $\theta_j = \theta_j' + \frac{\sigma}{1+\sigma}\mathbf{Q}_{ji}y$.
- Why?
  - For any realization of liquidity shocks $\{\theta_j\}_{j \neq i}$ we have $e_j = \theta_j - w_j(\theta_j)$.
  - If $\mathbb{S}_j$ is a cutoff strategy then:

$$e_j = \theta_j - F\left(\frac{s_j^* - \theta_j}{\sigma}\right)$$
$$= \frac{1+\sigma}{\sigma}\theta_j - \frac{s_j^* + \sigma/2}{\sigma}$$

    where $F(\cdot)$ is the conditional distribution of θ_j given s_j.
  - When $w_i = w_i^*$, then $x_{ij}(w_i^*) = \mathbf{Q}_{ji}y$ whereas when $w_i = w_i^0$, then $x_{ij} = 0$. In the latter case, by providing $\theta_j = \theta_j' + \frac{\sigma}{1+\sigma}\mathbf{Q}_{ji}y$, we see that:

$$\frac{1+\sigma}{\sigma}\left(\theta_j' + \frac{\sigma}{1+\sigma}\mathbf{Q}_{ji}y\right) - \frac{s_j^* + \sigma/2}{\sigma} = \frac{1+\sigma}{\sigma}\theta_j' - \frac{s_j^* + \sigma/2}{\sigma} + \mathbf{Q}_{ji}y$$

- But this only works if we have already assumed that $\{\mathbb{S}_j\}_{j \neq i}$ is a cutoff strategy that does not depend on $\mathbb{S}_i$!
- Clearly the reimbursement will change if we cannot assume the equilibrium withdrawal signal set $\mathbb{S}_j$ is in cutoff form and is independent of $\mathbb{S}_i$.
- I believe you can argue there is a reimbursement function $\theta_j^{reimburse}(\theta_j')$, but as far as I can tell, at this point we cannot prove it has any structure (and may depend on s_j^*).

Lemma 2 proof sketch, cont.

- Let's take the claim for now as given: $\sum_{j \neq i} x_{ij}(w_i^*)$ with $\theta_j = \theta_j'$ is the same as $\sum_{j \neq i} x_{ij}(w_i^0)$ when $\theta_j = \theta_j' + \frac{\sigma}{1+\sigma}\mathbf{Q}_{ji}y$. (But remember this reimbursement function depends on other banks' creditors using cutoff strategies.)
  - Thus:

$$\int_{\theta_j \in [\bar{\theta} - \frac{\sigma}{1+\sigma}\mathbf{Q}_{ji}y, \bar{\theta}]} \sum_{j \neq i} x_{ij}(w_i^*) dF(\theta_j | s_{im}) = \int_{\theta_j \in [\underline{\theta}, \underline{\theta} + \frac{\sigma}{1+\sigma}\mathbf{Q}_{ji}y]} \sum_{j \neq i} x_{ij}(w_i^0) dF(\theta_j | s_i)$$

  - We can therefore simplify the expression as:

$$\int_{\theta_j \in [\underline{\theta}, \bar{\theta}]} \frac{1+r}{1+\sigma} \left(\sum_{j \neq i} x_{ij}(w_i^*) - \sum_{j \neq i} x_{ij}(w_i^0) \right) dF(\theta_j | s_{im})$$

$$= \frac{1+r}{1+\sigma} \int_{\theta_j \in [\bar{\theta} - \frac{\sigma}{1+\sigma}\mathbf{Q}_{ji}y, \bar{\theta}]} \sum_{j \neq i} x_{ij}(w_i^*) dF(\theta_j | s_{im})$$

$$- \frac{1+r}{1+\sigma} \int_{\theta_j \in [\underline{\theta}, \underline{\theta} + \frac{\sigma}{1+\sigma}\mathbf{Q}_{ji}y]} \sum_{j \neq i} x_{ij}(w_i^0) dF(\theta_j | s_i)$$

  - Because $y \leq y_0 \equiv \min\{-\underline{\theta}, 1 - \bar{\theta}\}$, we know that $\bar{\theta} - \frac{\sigma}{1+\sigma}\mathbf{Q}_{ji}y > 1$ and $\underline{\theta} + \frac{\sigma}{1+\sigma}\mathbf{Q}_{ji}y < 0$.
  - This means that $x_j = y$ (full repayment from bank j) when $\theta_j \in [\bar{\theta} - \frac{\sigma}{1+\sigma}\mathbf{Q}_{ji}, y, \bar{\theta}]$ and $x_j = 0$ (complete default) when $\theta_j \in [\underline{\theta}, \underline{\theta} + \frac{\sigma}{1+\sigma}\mathbf{Q}_{ji}y]$.

Lemma 2 proof sketch, cont.

- Therefore, the second expression is zero, and first expression has $\sum_j x_{ij}(w_i^*) = y$, so we get:

$$\int_{\boldsymbol{\theta}_{-i}} \frac{1+r}{1+\sigma} \left(\sum_{j \neq i} x_{ij}(w_i^*) - \sum_{j \neq i} x_{ij}(w_i^0) - y \right) dJ(\boldsymbol{\theta}_{-i} | s_{im}) = \frac{1+r}{1+\sigma} y \cdot \prod_{j \neq i} \frac{\left(\frac{\sigma}{1+\sigma}\right) \mathbf{Q}_{ji} y}{\bar{\theta} - \underline{\theta}}$$

- In any (symmetric) network that is not complete, we have $\prod_{j \neq i} \mathbf{Q}_{ji} = 0$; in the complete network, it is "very small": if $\sigma \to 0$ or we assume $\bar{\theta} \to \infty$ and $\underline{\theta} \to -\infty$ (i.e., the improper uniform prior), it vanishes.
- All that remains is to show that $U(s_i^*, s_i^*)$ is increasing in s_i^*:

$$\frac{dU}{ds_i^*} = \frac{1+r}{1+\sigma} - \log \left(\frac{2}{\frac{s_i^* + \sigma/2 + \sum_{j \neq i} x_{ij}(w_i^0)}{1+\sigma} + 1} \right) \geq \frac{1+r}{1+\sigma} - \log(2)$$

  - Note that $s_i^* + \sigma/2 > 0$, because $s_i < -\sigma/2$ is the lower dominance region.
- It follows that $\frac{dU}{ds_i^*} \geq 0$ if $\sigma \leq \sigma_0 = (1+r)/\log(2) - 1$.
- This holds for *any* $\{\mathbb{S}_j\}_{j \neq i}$ and since $H(s_i^*, s_i^*)$ is increasing in s_i^* when $\sigma \leq \sigma_0$, there must be a unique value of s_i^* such that $H(s_i^*, s_i^*) = 0$.
- There is a unique equilibrium for all creditors of a bank i for a fixed $\{\mathbb{S}_j\}_{j \neq i}$, in a unique cutoff strategy s^*, which is the only rationalizable strategy.

Back to the reimbursement step...

- Recall $\theta_j^{reimburse}(\theta'_j)$ is the amount that bank $j \neq i$ needs such tha $x_{ij} = 0$ with θ_j instead of $x_{ij} = y$ with θ'_j yields the same leftover liquidity e_j.
  - This prevents the clearing vector $\mathbf{x}$ from changing.
- Without the assumption that $\mathbb{S}_j$ does not depend on $\mathbb{S}_i$, it is very possible that this reimbursement depends on s_i^*, i.e., we should write $\theta_j^{reimburse}(\theta'_j, s_i^*)$.
  - There may exist other equilibria, because the reimbursement function can mutually depend on others' cutoff strategies.
  - For example, high-cutoff equilibria at certain banks may entice high-cutoff equilibria at other banks.
  - Because of the uniform distribution and cutoff assumption, the lack of repayment does not interact with the cutoffs, only the liquidity shock.
- What the proof does show is that *there exists* an equilibrium in cutoff strategies, where these cutoffs do not depend on the cutoffs of creditors at other banks.
- Still an impressive claim, but much weaker than claiming there is a cutoff which is the only rationalizable strategy...

Existence and Uniqueness

Let's ignore this and press on (maybe I'm wrong?). If I am right, then we instead will prove the claim within the class of cutoff strategy equilibria where creditors are indifferent about other creditors' strategies.

Proposition

There exists a unique equilibrium in any regular and symmetric network. This equilibrium is symmetric, i.e., $s_i^* = s^* \in [\underline{s}, \overline{s}]$ *for all* i*.*

Proof Idea. Step 1: There exists a symmetric equilibrium, and among the class of symmetric profiles there is only one equilibrium.

- We know the strategy set of each bank j's creditors satisfy $\mathbb{S}_j = [\underline{s}, s_j^*)$ given *any* strategy of other banks $\{\mathbb{S}_\ell\}_{\ell \neq j}$;
- It can be shown that $H(s^*, s^*, \{\mathbb{S}_j = [\underline{s}, s^*)\}_{j \neq i})$ is increasing in s^* (intuitively, the game between creditors is of strategic complements when the difference in payoffs is positive);
- Given the dominance regions, there must exist an equilibrium cutoff where $H(s^*, s^*, \{\mathbb{S}_j = [\underline{s}, s^*)\}_{j \neq i}) = 0$ and it must be unique.

Existence and Uniqueness, cont.

Step 2: No asymmetric equilibria exist i.e., no bank can have the "highest" s_i^*.

- *Idea*: Bank i with the highest s_i^* better have the riskiest counterparties.
  - Naive proof: Bank i has the highest s_i^*, so necessarily is the riskiest, but counterparties are strictly less risky, by assumption.
  - Why it fails: Bank j_1 borrows from bank i and super safe bank j_2 (with s_{j_2} much lower than rest of banks); bank j_3 borrows from bank i and super safe bank j_4 (with s_{j_4} much lower than the rest of banks). Bank i can be very risky and $s_{j_1}^*$ and $s_{j_3}^*$ can still be lower than s_i^*.
- They show this to be true in a ring network (if s_1^* is largest, then bank 2 has the most counterparty exposure), and a complete network (if s_1^* is the largest, then every other bank has more counterparty exposure than bank 1);
- Sufficient to consider regular, circulant networks of degree 2 (why?):
  - Then $\mathbf{Q}_{i,i+1} = \mathbf{Q}_{n,1} = \alpha$ and $\mathbf{Q}_{i,i+2} = \mathbf{Q}_{n,2} = \mathbf{Q}_{n-1,1} = 1 - \alpha$, assume s_1^* is largest.
  - In the case $s_n^* \geq s_{n-1}^*$ bank 2's share of claims on bank 1 is α and its share of claims on bank n is $1 - \alpha$, but bank 1's share of claims on bank n is α and its share of claims on bank $n - 1$ is $1 - \alpha$. This implies $s_2^* \geq s_1^*$. (1 is worse than n and n is worse than $n - 1$.)
  - Suppose instead $s_{n-1}^* \geq s_n^*$. Bank n's counterparties are bank $n - 1$ and bank $n - 2$, who claim shares α and $1 - \alpha$, respectively, while bank 1's counterparties are bank n and bank $n - 1$, who claim shares α and $1 - \alpha$. It must be the case that $s_{n-2}^* \leq s_{n-1}^*$ in order to have $s_1^* \geq s_n^*$ ($n - 1$ is worse than n so $n - 2$ must be better than $n - 1$ for bank 1 to be most risky). But then $s_1^* \geq s_2^* \geq \ldots \geq s_n^*$, contradicting that bank 1 has the riskiest counterparties.

Fragility

Definition

A regular and symmetric financial network $(\mathbf{Q}^1, \mathbf{y}^1)$ is more *fragile* than $(\mathbf{Q}^2, \mathbf{y}^2)$ if for each bank i, the *ex-ante* probability of default is higher under $(\mathbf{Q}^1, \mathbf{y}^1)$ than under $(\mathbf{Q}^2, \mathbf{y}^2)$.

One can partially order financial networks by diversification by saying that $\mathbf{Q}^1$ is *more diversified* than $\mathbf{Q}^2$ (denoted $\mathbf{Q}^1 \succeq^d \mathbf{Q}^2$), if $\mathbf{Q}^1$ is a convex combination of $\mathbf{Q}^2$.

Proposition

Let $(\mathbf{Q}^1, \mathbf{y})$ and $(\mathbf{Q}^2, \mathbf{y})$ be symmetric and regular financial networks and suppose $\mathbf{y} \leq y_0 \mathbf{1}$:

1. *If $\mathbf{Q}^1 \succeq^d \mathbf{Q}^2$, then $s^*(\mathbf{Q}^1, \mathbf{y}) \geq s^*(\mathbf{Q}^2, \mathbf{y})$ so $\mathbf{Q}^1$ is more fragile than $\mathbf{Q}^2$;*
2. *The ring network is the least fragile symmetric network and the complete network is most fragile symmetric network;*
3. *Out of all the financial networks in which each bank connects to m banks in a symmetric network of n banks, the network with the most evenly distributed connections is the most fragile.*

Example from the paper

- Compare the ring and complete network: assume that $\sigma \to 0$ and $y < \min\left\{\frac{1}{1+r}, y_0\right\}$.

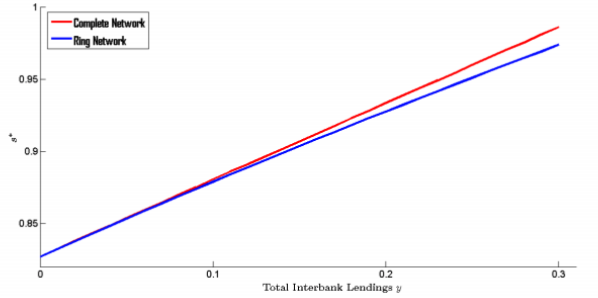


Figure 5: Equilibrium Threshold: the Ring and Complete Network

- The diversification effect then intensifies panic is increasing in the size of the total interbank lending y.

Information Disclosure

- Suppose creditors do not have information about the connections of their bank; in particular, let some bank ℓ have a bad liquidity shock θ_ℓ, but the network is symmetric and the source of the bad liquidity shock is unknown.
- The liquidity shocks are also such that $\theta_i \in U[\underline{\theta}, \overline{\theta}]$ but

$$\theta_j = \begin{cases} \theta_\ell, \text{w.p. } \frac{1}{n-1} \\ \sim U[\underline{\theta}, \overline{\theta}], \text{w.p. } \frac{n-2}{n-1} \end{cases}$$

- Claims the existence/uniqueness results will carry over to this model immediately (why?)
- Similar results hold: the symmetric and regular network $(\mathbf{Q}_1, \mathbf{y})$ is more fragile than $(\mathbf{Q}_2, \mathbf{y})$ if $\mathbf{Q}_1$ has more diversified connections than $\mathbf{Q}_2$.
- Now assume linkages of the network are complete information; in particular, a bank is aware whether it is connected to bank ℓ.

Proposition

When the information about the (initially) distressed bank is available, there exists $n_0 > 0$ such that the ring network consisting of $n < n_0$ banks is more fragile than the complete network of n banks.

- Panic intensifies when the contagion is extreme (less diversification) and the exact source of this contagion is known.